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Existence, Extendability, and Stability for a Differential Equation in the Space of Convergent Sequences

S. R. BERNFELD

Department of Mathematics, University of Texas, Arlington, Texas 76019

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1. INTRODUCTION

Two independent, but not unrelated, thoughts motivated the development of the example (and its variations) presented in this note. The setting for the example is the Banach space of convergent sequences and conditions are given on each component which either force the component to vanish in a finite time or "allow" it to exist in the future. In fact the behavior of each component depends not only on itself but also on those components proceeding it. That is if a_I is an arbitrary given component of the element $a = \{a_i\}_{i=1}^{\infty}$ then the Ith component of the solution $u(t, a)$, $u_I(t, a)$, depends on all the components $u_i(t, a)$ for which $i \geqslant I$. Moreover each of the $u_i(t, a)$ depends only on those components a_j for which $j \geqslant i \geqslant I$. Here $u(t, a)$ is the solution of the example in which $u(0, a) = a$.

The applicable motivation for the example is to consider a process consisting of a large number of compartments in series. Assume each compartment contains a certain amount of food, chemicals, or stored energy which changes with time, and the amount of material in each compartment obeys a modified first-order decay law. Because the number of compartments may be very large (and not known) and because there may be several such independent processes occurring, we find it expedient to imbed the problem into an infinite-dimensional setting, the space of vectors having an infinite number of components with each component representing a different compartment and each vector a different process. This appears more practical than using R^n as we do not know what n is except that it's very large. Some of the questions we address include the length of time that the process "lives," which depends on the amount of variation of a (roughly, we measure quantities like a_i/a_{i-1}), and in those cases for which the process continues on forever we look at questions of stability and exhibit appropriate Liapunov functions.

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The second motivation was to contrast the qualitative behavior of the solutions of (3.1) with those in the analogous finite-dimensional setting. Recall that Dieudonné [1] gave examples in a Banach space in which the Cauchy problem may not have a solution or it may not extend. See also Ladas and Lakshmikantham [2] for a treatment of this as well as related matters including relevant references. In our example, which is different in spirit than those given in [1, 2], we find there are initial points a for which there is no solution; essentially the variation of a becomes unbounded as we traverse its components. There are other points a for which we have only finite extendability and there are still other points through which solution exists forever. Discussions of local and global invariant sets, continuous dependence, stability, and asymptotic stability highlight some differences between the behavior of solutions of (3.1) and that in R^n. Particular emphasis is made concerning the stability of the solution $u(t, a) \equiv 0$ which lies on the boundary of our set; then as we will see we do not have and thus do not require existence or extendability of solutions in a neighborhood of $u(t, a) \equiv 0$ in order to discuss stability or Liapunov functions associated with the stability (see Lakshmikantham [3] and the notes of L. Salvadori [4] concerning this in a general autonomous dynamical system).

2. Preliminaries

Let E be the Banach space of real sequences, $u = \{u_i\}_{i=1}^{\infty}$, which converge and for $u \in E$ define $\|u\| = \sup_{1 \leqslant i < \infty} |u_i|$. Denote by E_c those sequences which converge to c. Let $E^+ = \{u \in E: u_i \geqslant 0\}$ and $E^- = \{u \in E: u_i \leqslant 0\}$ and similarly define E_c^+ and E_c^-. Denote by B the set of elements $u \in E$ for which there exist integers J, K with $J < K$ such that $u_J = 0$ and $u_K \neq 0$ and define $B^+ = B \cap E^+$, $V_c = B \cap E_c$, $B_c^+ = B \cap E_c^+$ (similar definitions hold for B^-, B_c^-). Let $A = E \backslash B$ and as before define $A_c = A \cap E_c$, $A^+ = A \cap E^+$, etc. If C is any set denote by $\bar{C}$ the closure of C; int C, the interior of C; and ∂C, the boundary of C.

For each $u \in E$ define $u^J \in E$ to be that element for which $u_i^J = 0$, $1 \leqslant i \leqslant J-1$, and $u_i^J = u_i$, $i \geqslant J$. Define h: $E \to E$ as

$$h_j(u) = -\|u^j\| \quad \text{whenever} \quad u_j \geqslant 0$$
$$= -u_j \quad \text{whenever} \quad u_j < 0.$$

We shall be analyzing the differential equation $\dot{u} = h(u)$ and in order to do so we will look at the largest subset of E for which h is continuous, namely, A. Indeed let $u \in B_j$ then there exist integers J and K, $J < K$, such that $u_J = 0$ and $u_K \neq 0$. Now consider a sequence of points $a(n) = \{a_i(n)\}_{i=1}^{\infty} \in E$, $n = 1, 2, \ldots$, for which $a_J(n) > 0$ for all n, $a_J(n) \to 0$ as $n \to \infty$, and $a(n) \to u$ as $n \to \infty$. Similarly consider a sequence $b(b) \in E$ such that $b_J(n) < 0$ for all

n, $b_J(n) \to 0$ as $n \to \infty$, and $b(n) \to u$ as $n \to \infty$. By definition $h_J(b(n)) = -b_J(n) \to 0$ as $n \to \infty$. However, for n sufficiently large $h_J(a(n)) = -\|a^J(n)\|$ and thus $\lim_{n\to\infty} h_J(a(n)) = -\lim_{n\to\infty} \|a^J(n)\| = -\|u^J\| \leqslant -\|u^K\| < 0$. Consequently h is not continuous on B. We now show h is continuous on A. Take any element $a \in A$. Clearly $h(a) \in A$. There are two cases to consider: (i) $a \in A_c$, $c \neq 0$ or (ii) $a \in A_0$. Case (i) follows easily from the definition of h since $a_i \neq 0$ for all i and whenever $b(n) \to a$ as $n \to \infty$ then for all i the sign of $b_i(n)$ is the same as the sign of a_i for sufficiently large n. In case (ii) either there exists an integer J for which $a_k \neq 0$ for $k \leqslant J$ and $a_k = 0$ for $k > J$ or $a_k \neq 0$ for all k. Let $b(n) \to a$ as $n \to \infty$; then for $\varepsilon > 0$ there exists N_1 such that $\|b_j(n) - a_j\| < \varepsilon/4$ for $n > N_1$ and for all j. Since $a \in A_0$ there exists $J_1 \geqslant J$ such that $\|a^j\| < \varepsilon/4$ for $j > J_1$. Thus for $h > J_1$ and $n > N_i$, $\|b^j(n)\| < \varepsilon/2$ and this implies $|h_j(b(n)) - h_j(a)| \leqslant \|b^j(n)\| + \|a^j\| < \varepsilon$. In the case where $a_k \neq 0$ for all k we have the existence of N_2 such that $b_j(n)$ has the same sign as a_j for $n \geqslant N_2$ and $j \leqslant J_1$. This implies that there exists $N_3 \geqslant N_2$ such that $|h_j(b(n)) - h_j(a)| < \varepsilon$ for $j \leqslant J_1$ and $n \geqslant N_3$. Thus for $n \geqslant \max(N_3, N_1)$, $\|h(b(n) - h(a)\| < \varepsilon$, and the case where $a_k \neq 0$ for $k \leqslant J$ and $a_k = 0$ for $k \geqslant J$ we can use the same argument, where now $J_1 = J$, to conclude that $\|h(b(n)) - h(a)\| < \varepsilon$ for n sufficiently large. Hence h is continuous on A. It is also not difficult to see that h is locally Lipschitz in the int A.

We now show $A_0 = \partial A \cap A$ and that A is neither open nor closed. Let $a \in A_0$, then for each integer n there exists $J(n)$ such that for $j \geqslant J(n)$, $|a_j| < 1/n$. Choose a sequence $\{b(n)\}_{n=1}^{\infty} \in B$ such that for all n, $b_i(n) = a_i$ for $i \neq J$, $J+1$, $b_J(n) = 0$, $b_{J+1}(n) \neq 0$, and $|b_{J+1}(n) - a_{J+1}(n)| < 1/n$. Since $\|b(n) - a\| < 1/n$, $A_0 \subset \partial A$. Suppose $a \in A \backslash A_0$, that is, $a \in A_c$, $c \neq 0$. We claim a is in the interior of A. Suppose not, then there exists a sequence $\{b(n)\}_{n=1}^{\infty}$ contained in B such that $b(n) \to a$ as $n \to \infty$. For each n there exists $N(n)$ such that $|b_j(n)| > |c|/2$ for $j > N$; hence there exists a bounded set of integers $J(n)$ for which $b_{J(n)}^{(n)} = 0$. Without loss of generality we may assume $J(n) \to \bar{J}$ implying that $a_{\bar{J}} = 0$. Since $a \in A_c$ there exists an integer $J_1 > \bar{J}$ for which $|a_{J_1}| \neq 0$, a contradiction to the fact that $a \notin B$. Hence $A_0 = \partial A \cap A$ or int $A = A \backslash A_0 = \bigcup_c A_c$, $c \neq 0$; moreover $\bar{A} = \overline{\operatorname{int} A} = E$.

3. Example in $A^+ \cup A^-$, Asymptotic Stability

Let us now consider the ordinary differential equation $\dot{u} = h(u)$ defined on A; that is, for $a \in A$ consider the Cauchy problem

$$\begin{aligned} \dot{u}_j &= -\|u^j\| \qquad \text{if} \quad u_j \geqslant 0 \\ \dot{u}_j &= -u_j \qquad \text{if} \quad u_j < 0 \\ u(0) &= a. \end{aligned} \tag{3.1}$$

By a solution $u(\cdot, a)$ of (3.1) we shall mean that there exists a $T>0$, T finite or infinite, and a function $u(\cdot, a)$: $[0, T) \to A$ satisfying (2.1 on $[0, T)$. Sometimes we will omit the initial point a and represent the solution as $u(t)$. We say a solution $u(\cdot)$ is a nonextendable solution if $T<\infty$ and $\lim_{t\to T^-} u(t) \notin A$ (we show that $\lim_{t\to T^-} u(t)$ always exists in E and will refer to T as the exit time). When $T=\infty$ we will call the solution $u(\cdot)$ a global solution. The interval $[0, T)$ in these two cases will be called the domain of existence of $u(\cdot)$. A subset $K \in A$ is said to be locally invariant if for each $a \in K$ the solution $u(\cdot)$ is contained in K on its domain of existence. If in addition $u(\cdot)$ is a global solution for each $a \in K$ and is contained in K on $[0, \infty)$ then K is said to be globally invariant.

Clearly for any $a \in A^-$ the solution $u(t)$ is a global one in which $\|u(t)\| \to 0$ as $t \to \infty$. Let $a \in A \backslash A^-$ and first consider the case in which $a \in A_0^+$. Define inductively $a_{K_{i+1}} = \sup_{j>K_i}\{a_j\}$, $i=0, 1, 2, \ldots$ (since K_{i+1} is not necessarily unique pick the maximum one), where $K_0=1$; clearly the set $\{K_i\}$ is infinite. Define the intervals $I_i=[K_{i-1}, K_i]$, $i=1, 2, \ldots$. Then $I_i \cap I_{i+1} = K_i$ and we refer to $\{I_i\}_{i=1}^{\infty}$ as an infinite decomposition of a. We say I_i is inadmissible if $K_i - K_{i-1} > 1$ and there exists $n_i \in I_i$ such that $a_{n_i} < a_{K_i}$ (we call a_{n_i} an inadmissible point). Pick any K_i and any inadmissible integer n_i contained in I_i; then $a_{K_i} > a_{n_i}$. Since $a_{n_i} > 0$ then from (3.1) there exists $t_{n_i} > 0$ for which $u_{K_i}(t) > u_{n_i}(t) > 0$ for $t \in [0, t_{n_i})$. In fact we show $0 < t_{n_i} < \infty$ and $u_{n_i}(t_{n_i}) = 0$. Indeed since $a_{K_i} > a_J$ for every integer $J > K_i$ then there exists an interval $[0, T_J)$, $T_J > 0$, for which $u_{K_i}(t) > u_J(t) > 0$ on $[0, T_J)$. Then either $\inf_{J \geq K_i} T_J = 0$, in which case no solution of (3.1) exists, or $\inf_{J \geq K_i} T_J \geq T > 0$. In the latter case we have for $t \in [0, T)$ that $\dot u_{K_i}(t) = -\|u^{K_i}(t)\|^i = -u_{K_i}(t)$ since $0 < u_J(t) < u_{K_i}(t)$. Hence for $t \in [0, T)$, $u_{K_i}(t) = a_{K_i} e^{-t}$ and $\dot u_{n_i}(t) = -\|u^{K_i}(t)\| = -u_{K_i}(t) = -a_{K_i} e^{-t}$; hence $u_{n_i}(t) = a_{n_i} + a_{K_i}(e^{-t} - 1) = a_{n_i} - a_{K_i} + u_{K_i}(t)$. Thus for as long as $u_{n_i}(t) > 0$, $u_{K_i}(t) - u_{n_i}(t) \equiv u_{K_i}(0) - u_{n_i}(0) = a_{K_i} - a_{n_i} > 0$. In fact $u_{n_i}(t) > 0$ on $[0, t_{n_i})$, $u_{n_i}(t_{n_i}) = 0$, where $t_{n_i} = -\ln(1 - a_{n_i}/a_{K_i})$, implying $u(t_{n_i}) \notin A$ since $u_{k_i}(t_{n_i}) > 0$. We refer to the time t_{n_i} as the exit time for the component $u_{t_{n_i}}(t)$ of the solution $u(t)$. This analysis shows that if each I_i is admissible (that is, not inadmissible) then for each K_i, $u_{K_i}(t) = a_{K_i} e^{-t}$, thus implying $u(t) = ae^{-t}$ since for each integer j there exists K_j such that $a_{K_j} = a_j$. Hence $u(t)$ is a global solution and $\|u(t)\| \to 0$ as $t \to \infty$ at an exponential rate. This case occurs if and only if $a_{i+1} \leq a_i$, that is, a is nonincreasing. Assume, on the other hand, that there exist a finite number N of inadmissible intervals $\{\tilde I_i\}_{i=1}^N \subset \{I_i\}_{i=1}^{\infty}$. Denote $\tilde I_i = [\tilde K_{i-1}, \tilde K_i]$, where $\{\tilde K_i\}_{i=0}^N \subset \{K_i\}_{i=0}^{\infty}$. Define $a_{s_i} = \min_{a_j \in I_i}\{a_j\}$ and call $a_{\tilde K_i}/a_{s_i}$ the upward slope of a in $\tilde I_i$. The corresponding component $u_{s_i}(t)$ has exit time $t_{s_i} = -\ln(1 - a_{s_i}/a_{\tilde K_i})$. Notice that the exit time of the other components in $\tilde I_i$ is greater than or equal to t_{s_i} since $a_{r_i}/a_{\tilde K_i} \geq a_{s_i}/a_{\tilde K_i}$ for every other component a_{r_i} in $\tilde I_i$, implying $t_{r_i} \geq t_{s_i}$. If we define $T = \min_{1 \leq i \leq N} t_{s_i}$ then $T>0$ is the exit time for $u(t)$. Hence

$u(t)>0$ for $t\in[0,T)$ with $\lim_{t\to T^-} u(t)=u(T)\notin A$. Notice that in the admissible intervals no component vanishes and thus these intervals do not contribute to the calculation of the exit time for $u(t)$. If there are an infinite number of inadmissible intervals $\{\tilde{I}_i\}_{i=1}^{\infty}$, where $\tilde{I}_i=[\tilde{K}_{i-1}, \tilde{K}_i]$, then we say a oscillates (call this set O). As before let $a_{\tau_i}=\min_{a_j\in \tilde{I}_i}\{a_j\}$ be the minimal inadmissible point in $\tilde{I}_i$ and let t_{τ_i} be the exit time of the component $u_{\tau_i}(t)$. Then a solution $u(t)$ exists (and is nonextendable) if and only if the exit time $T\equiv\inf_{1\leqslant i\leqslant\infty} t_{\tau_i}>0$. Otherwise no solution exists if and only if $T=\underline{\lim}_{i\to\infty} t_{\tau_i}=0$. Since $t_{\tau_i}=-\ln(1-a_{\tau_i}/a_{\tilde{K}_i})$ then $T=\underline{\lim}_{i\to\infty} -\ln(1-a_{\tau_i}/a_{\tilde{K}_i})$ and thus $T=0$ if and only if $\underline{\lim}_{i\to\infty}(a_{\tau_i}/a_{\tilde{K}_i})=\inf_{1\leqslant i<\infty}(a_{\tau_i}/a_{\tilde{K}_i})=0$. In this case we say that a oscillates with unbounded upward slopes and thus no solution exists. Thus $u(t, a)$ is a nonextendable solution if and only if $\inf_{1\leqslant i<\infty}(a_{\tau_i}/a_{\tilde{K}_i})>0$, that is, if the upward slope is bounded. Hence the exit time of $u(t, a)$ is a measure of the upward slope of a.

Let us now suppose $a\in A_c^+$ for some $c\neq 0$. We now construct a decomposition of $[0, \infty)$, but in contrast to the previous case there may not exist an infinite decomposition unless we impose additional constraints. In fact there may only be a finite decomposition, that is, there exist $K_1, ..., K_N$ such that $[1, \infty)=\bigcup_{i=1}^{N}[K_{i-1}, K_i]\cup[K_N, \infty)$. Also there is no decomposition if and only if a is nondecreasing, that is, $a_{i+1}\geqslant a_i$ for all i since $K_0=1$ and K_1 does not exist. We say $N=0$ in this case. Moreover there is a finite decomposition if and only if a is eventually nondecreasing; that is, there exists an integer P in which $a_{i+1}\geqslant a_i$ for $i\geqslant p$ and there exists an integer $r<p$ in which $a_r<a_{r+1}$. First assume there is an infinite decomposition; as before, if there are no inadmissible intervals then $u(t)$ is a global solution and $\|u(t)\|=a_{\bar{K}}e^{-t}$ where $\bar{K}=\max(K_0, K_1)$. Moreover if there are a finite number of inadmissible intervals then as before $u(t)$ is a nonextendable solution and the exit time can be computed exactly as before. If there are an infinite number of inadmissible intervals, that is, a oscillates, then, in contrast to the case when $a\in A^+$, we find (using the same notation as before) that $\underline{\lim}_{i\to\infty} t_{\tau_i}=-\lim_{i\to\infty}\ln(1-a_{\tau_i}/a_{\tilde{K}_i})\neq 0$. Indeed $\lim_{i\to\infty}(a_{\tau_i}/a_{\tilde{K}_i})=1$ since $\lim_{i\to\infty} a_{\tau_i}=\lim_{i\to\infty} a_{\tilde{K}_i}=c\neq 0$ and consequently $T=\inf_{i\in I_i} t_{\tau_i}>0$, where T is the exit time of $u(t)$. Hence a cannot oscillate with unbounded upward variation; therefore $u(t, a)$ is a nonextendable solution. Now assume there is no decomposition, that is, a is nondecreasing. Then for each i, $a_i\leqslant c$, in fact without loss of generality assume $a_i<c$. Then there exists $T>0$ such that $0<u_i(t)\leqslant u_{i+1}(t)$ for $t\in[0, T)$ since $0<a_1\leqslant a_i$ for all $i>1$. Moreover $\dot{u}_{i+1}(t)=\dot{u}_i(t)$ for each i, implying $u_{i+1}(t)=u_i(t)+a_{i+1}-a_i$. Since $\dot{u}_i(t)\leqslant -u_{i+1}(t)=-u_i(t)+a_i-a_{i+1}$ then $u_i(t)\leqslant a_i e^{-t}-(a_{i+1}-a_i)(1-e^{-t})$ on $[0, T)$. Thus $u_{i-1}(t)\leqslant u_i(t)\leqslant a_{i+1}e^{-t}-(a_{i+1}-a_i)$; that is, $u_i(t)$ vanishes at some $\bar{t}_i\leqslant -\ln(1-a_i/a_{i+1})$. In fact we find $\bar{t}_i=-\ln(1-a_i/c)$ since $u_i(t)=ce^{-t}+a_i-c$. Hence $u(t)$ is a nonextendable solution defined on an interval $[0, T)$, where $0 < T = \inf_i(-\ln(1-a_i/c)) = -\ln(1-a_1/c)$ (here T is the exit time). Finally, in the case where there is a finite decomposition we combine the previous analysis to obtain that $u(t)$ is a nonextendable solution defined on $[0, T)$. Indeed there exists $p > 0$ in which a_i is nondecreasing for $i \geqslant p$; on the interval $[0, p]$ there are a finite number of intervals. If none of these intervals is inadmissible then $T = -\ln(1-a_p/c)$. If there exist $J \geqslant 1$ inadmissible intervals $\{\bar{I}_i\}_{i=1}^{J}$ then in each interval define τ_i, $0 < \tau_i < \infty$, as $a_{\tau_i} = \inf_{j \in I_i}\{a_j\}$. Then let t_{τ_i} be the exit time for the component $u_{\tau_i}(t)$ of the solution $u(t)$. For any other integer $\bar{i} \in \bar{I}_i$ there exists $s_{\bar{i}} \geqslant s_i$ for which $u_i(t) > 0$ for $t \in [0, s_i)$. In this case $T = \min_{i \in \bar{I}_i}\{s_i, -\ln(1-p/c)\}$.

Let us now summarize what we have obtained in the following proposition.

PROPOSITION 3.1. I. *Let $a \in A_0^+$. Then we have the following three sets of equivalent statements concerning* (2.1):

A. (i) *$u(t,a)$ is a global solution and then $\|u(t)\| \to 0$ as $r \to \infty$,*

(ii) *a is nonincreasing (define NI to be those elements a in A which are nonincreasing; let $NI^+ = NI \cap A^+$, $NI_0^+ = NI \cap A_0^+$);*

B. (i) *$u(t, a)$ is a nonextendable solution and then there exists the exit time $T > 0$ such that $u(t) \in A$ for $t \in [0, T)$ but $u(T) \notin A$,*

(ii) *a is eventually nonincreasing ($EVNI^+$), or a oscillates with bounded upward slope (OB_0^+);*

C. (i) *there is no solution,*

(ii) *a oscillates with unbounded upward slope.*

II. *Let $a \in \bigcup_{c \neq 0} A_c^+ = \operatorname{int} A^+$. In this case all solutions exist. Then we have the following two sets of equivalent statements concerning* (2.1):

A. (i) *$u(t, a)$ is a global solution and then $\|u(t, a)\| \to 0$ and $t \to \infty$,*

(ii) *a is nonincreasing, (NI^+);*

B. (i) *$u(t, a)$ is a nonextendable solution and then there exists the exit time $T > 0$ such that $u(t) \in A$ for $r \in [0, T)$ but $u(T) \notin A$,*

(ii) *a is eventually nonincreasing ($EVNI^+$), or a is eventually nondecreasing ($EVND^+$), or a is nondecreasing, and not a constant (ND^+), or a oscillates (with bounded upward slope) ($\bigcup_{c \neq 0} UB_c^+$).*

III. *Let $a \in A^-$; then $u(t, a)$ is a global solution such that $\|u(t)\| \to 0$ as $t \to \infty$.*

We shall now look at the invariant sets as well as further asymptotic properties of these.

It is convenient to give a measure of the "total upward variation" of a, call it $m(a)$, in terms of its decomposition. When I_i is an admissible interval define the upward variation of a on I_i, denoted by $m_i(a)$, as $m_i(a)=1$. When the decomposition of a contains inadmissible intervals $\tilde{I}_i$ then consider the following situations: First suppose $a\in A^+$ such that a is not nondecreasing (from Proposition 3.1, a can't be nonincreasing). Then the decomposition of a contains (using previous notation) $N\geqslant 1$ inadmissible intervals $\{\tilde{I}_i\}_{i=1}^N$, where $\tilde{I}_i=[\tilde{K}_{i-1}, \tilde{K}_i]$ and $1\leqslant N\leqslant\infty$. We say a finitely oscillates if $N<\infty$ (recall a oscillates if $N=\infty$). As before let $a_{\tau_i}=\inf_{i\in\tilde{I}_i}\{a_i\}$. We define the measure of the upward variation of a on $\tilde{I}_i$, $m_i(a)$, and $m(a)$ as follows: (i) if $a\in A_0^+$ then $m_i(a)=a_{\tilde{K}_i}/a_{\tau_i}$ and $m(a)=\sup_{1\leqslant i<\infty} m_i(a)=\sup_{1\leqslant i\leqslant\infty}(a_{\tilde{K}_i}/a_{\tau_i})$; (ii) if $a\in A_c^+$ and $N=\infty$ then again $m(a)=\sup_{1\leqslant i\leqslant\infty}(a_{\tilde{K}_i}/a_{\tau_i})$; (iii) if $a\in A_c^+$ and $1\leqslant N<\infty$ then $m(a)=\max\{\max_{1\leqslant i\leqslant N}(a_{\tilde{K}_i}/a_{\tau_i}, c/a_{\tilde{K}_N}\}$. If a is nondecreasing (hence $a\in A_c^+$ for some $c\neq 0$) then $N=0$ and we define $m(a)=c/a_1$. Notice $m_i(a)=1$ if and only if I_i is admissible and $m_i(a)>1$ if and only if $\tilde{I}_i$ is inadmissible. Thus if a contains inadmissible intervals $m(a)=\sup_{i\in I_i} m_i(a)=\sup_{i\in\tilde{I}_i} m_i(a)$. Thus in general if $\{I_i\}_{i=1}^N$ is a decomposition of a then $m(a)=\sup_{i\in I_i} m_i(a)$, where $m_i(a)\geqslant 1$. Hence $u(t, a)$ is a global solution if and only if $m(a)=1$. In the case $u(t, a)$ is a nonextendable solution, the exit time $T(a)=\ln(m(a)/(m(a)-1))$. Thus T is a nonincreasing continuous function of m such that $T\searrow 0$ as $m\nearrow\infty$ (no solution) while $T\nearrow\infty$ as $m\searrow 1$ (global solution). In this way we consider $m(\cdot)$ to be an extended function with range $[0, \infty]$. We summarize this in the following proposition.

PROPOSITION 3.2. *Let $a\in A^+$ and let $m(a)$ be the measure of the upward slope of a defined above. Then*

(i) *$u(t, a)$ is a global solution if and only if $m(a)=1$,*

(ii) *$u(t, a)$ does not exist if and only if $m(a)=\infty$,*

(iii) *$u(t, a)$ is a nonextendable solution if and only if $1<m(a)<\infty$, whose exit time $T(a)=\ln(m(a)/(m(a)-1))$.*

In particular T is a strictly decreasing function of m in which $T\to 0$ as $m\to\infty$ and $T\to\infty$ as $m\to 1$.

(iv) *$m(u(t, a))$ is strictly increasing in t on $[0, T)$ when $1<m(a)<\infty$.*

We shall now look at the invariant subsets of $A^+\cup A^-$. Notice if $a\equiv 0$ then $u(t, \alpha)\equiv 0$ for all t so the point $a\equiv 0$ is globally invariant. Also A^- is globally invariant so we restrict our attention to A^+. From Proposition 3.1, int $A^+\cup NI^+$ is globally invariant and is in fact the largest globally invariant subset in the int A^+. Indeed if $a \in \text{int } A^+$ and a is nonincreasing then since $\dot{u}_j(t) = -\|u^j(t)\| = -u_j(t)$ it follows (as before) that $u_j(t) = a_j e^{-t}$. Hence for each $r \in [0, \infty)$, $u_j(t) \in \text{int } A^+$ and $u_j(t)$ is nonincreasing, that is, contained in NI^+. The fact that int $A^+ \cap NI^+$ is the largest globally invariant subset of int A^+ follows from Proposition 3.1, parts I.A, and II.A. Similarly $A_0^+ \cap NI^+$ is the largest globally invariant subset of $A^+ \cap \partial A^+$.

We now direct our attention to the locally invariant subsets (and from now on which are not globally invariant) of A^+. We need not consider A since there are no locally invariant subsets of A^- which are not also globally invariant. First assume $a \in \text{int } A^+$ and $a \notin NI^+$, that is, $T(a) < \infty$ or equivalently $m(a) > 1$. Then $a \in \bigcup_c A_c^+$, $c \neq 0$, and int $A^+ \backslash NI^+$ is locally invariant since $u(t, a) \in \bigcup_{c_t} A_{c_t}^+$ for each $t \in [0, T)$, where $0 < c_t = ce^{-t}$. Those subsets of int A^+ which are locally invariant may be found by utilizing Proposition 3.1, part II.B(II). In fact our previous analysis implies that each of the disjoint sets $S_1 \equiv \text{int } A^+ \cap EVNI^+$, $S_2 \equiv \text{int } A^+ \cap EVND^+$, $S_3 \equiv \text{int } A^+ \cap ND^+$, and $S_4 = \text{int } A^+ \cap OB^+$ is locally invariant. Moreover S_i is not uniformly locally invariant in that there exists no $T > 0$ such that for each $a \in S$, $u(t, a)$ is in S for $t \in [0, T)$. Also there are not other locally invariant subsets contained in the int A^+ which are disjoint from all of the $\{S_i\}_{i=1}^4$. In fact the only open locally invariant subsets of S_i are of the form $S_i \cap \{a: \|a\| < \beta\}$, where β is any positive number. We summarize this as follows:

Proposition 3.3. (i) *The sets A^- and* int A^+ *are globally invariant.*

(ii) *The* int $A^+ \cap NI^+$ *is the largest globally invariant subset of the* int A^+ *(all other globally invariant subsets of* int A^+ *are in NI^+).*

(iii) *The set $A_0^+ \cap NI^+$ is the largest globally invariant subset of $A^+ \cap \partial A^+$ (all other globally invariant subsets of A_0^+ are in NI^+). This set includes the origin.*

(iv) *The disjoint sets* int $A^+ \cap EVNI^+$, int $A^+ \cap EVND^+$, int $A^+ \cap ND^+$, *and* int $A^+ \cap OB^+$ *are locally invariant. All other locally invariant subsets in the* int A^+ *are contained in one of the S_i.*

We now shall look at the behavior of solutions near certain invariant sets as well as continuous dependence and stability behavior. Let $a \in \text{int } A^+$ and since $T(a)$, the exit time, is continuou with respect to a then $u(t, a)$ is continuous with respect to a. Indeed for b close to a consider the decomposition of b and a, $\{I_i^b\}_{i=1}^{N_1}$, $\{I_i^a\}_{i=1}^{N_2}$, where $1 \leqslant N_1 \leqslant \infty$ or $1 \leqslant N_2 \leqslant \infty$ or $N_1 = 0$ or $N_2 = 0$, such that $I_i^a = [K_{i-1}^a, K_i^a]$ and $I_i^b = [K_{i-1}^b, K_i^b]$. Denote $a_{K_i^a}$ by A_i, and $a_{K_i^b}$ by B_i. Given $\varepsilon > 0$ choose b such that $\|b - a\| < \varepsilon$, where $\{b: \|b - a\| < \varepsilon\}$ is contained in the int A^+. Then for each i,

$u_i(t, b) - u_i(t, a) = u_i(t, B_i) + b - B_i - u_i(t, A_i) - a + A_i$ or $|u_i(t, b) - u_i(t, a)| \leqslant |u_i(t, B_i) - u_i(t, A_i)| + \|b-a\| + |A_i - B_i|$, that is,

$$|u_i(t, b) - u_i(t, a)| \leqslant |B_i - A_i|(1 + e^{-t}) + \|b - a\|$$
$$\leqslant 3\|b-a\|,$$

or

$$\|u(t, b) - u(t, a)\| \leqslant 3\ \|b - a\| \leqslant 3\varepsilon.$$

For ε sufficiently small than $\hat{T} = \inf_{b \in S_b} T(b) > 0$, where $S_b = \{b: \|b-a\| < \varepsilon\}$. Hence our above calculations show that $u(t, a)$ is locally Lipschitz in a and that for each sufficiently small neighborhood $\mathcal{N}$ of a there is a time depending only on $\mathcal{N}$ for which all solutions exist. Notice also that for each $\alpha \in \operatorname{int} A^+$ for which there exists a global solution and in each neighborhood of a there exist points through which there are non-extendable solutions. This follows from Proposition 3.3(ii) since in any neighborhood of NI^+ we can construct an a which oscillates.

On $A^+ \cap \partial A^+ = A^+$ continuous dependence results differ from the previous case. For example, consider an element $a \in A_0^+ \cap NI^+$; then $T(a) = \infty$, that is, $u(t, a)$ is a global solution. Now consider the following sequence of points $\{b^n\}$ in A_0^+ for which $b^n \to a$ as $n \to \infty$, yet $T(b^n) = 0$, that is, $u(t, b^n)$ does not exist. Indeed first let $\alpha \equiv 0$ and define b^n as

$$b^n_{2j} = 1/2nj, \qquad j = 1, 2, \ldots$$
$$b^n_{2j-1} = 1/n(2j-1)^2, \qquad j = 1, 2, \ldots.$$

Clearly $b^n \to 0$ as $n \to \infty$. Since $[2j-1,\ 2j]$ are inadmissible intervals, it is easy to see that

$$m(b^n) = \sup_j \frac{1/2nj}{1/n(2j-1)^2}$$
$$= \lim_{j \to \infty} \frac{(2j-1)^2}{2j} = \infty,$$

This implies $T(b^n) = 0$. Hence in every neighborhood of $a = 0$ there exist points for which there exists no solution. Consequently the continuous dependence on initial conditions does not hold. Now let a be the "strictly decreasing" point (that is, $a_{j+1} < a_j$) given by $a_{2j} = 1/(2j)^2$, $j = 1, 2, \ldots$, $a_{2j-1} = 1/(2j-1)^2$, $j = 1, 2, \ldots$, and let b^n be defined as $b^n_{2j} = a_{2j} + 1/2nj$, $j = 1, 2, \ldots$, $b^n_{2j-1} = a_{2j-1} + 1/n(2j-1)^3$, $j = 1, 2, \ldots$. Clearly $b^n \to a$ as $n \to \infty$.

Also b^n oscillates; in fact since $[2j-1, 2j]$ are inadmissible intervals for j sufficiently large (depending on n) then

$$m(b^n) = \lim_{j\to\infty} \left(\frac{1}{4j^2} + \frac{1}{2nj}\right) \Big/ \left(\frac{1}{(2j-1)^2} + \frac{1}{n)2j-1)^3}\right) = \infty.$$

Hence $T(b^n) = 0$ and thus in every neighborhood of a there exist points through which there are no solutions. In fact for each $a \in A_0^+ \cap NI^+$ we can find a sequence b^n in A_0^+ which oscillates with unbounded upward slope such that $b^n \to a$, as $n \to \infty$ but $T(b^n) \equiv 0$ and $T(a) = \infty$. These arguments go through if $a \in A_0^+ \cap EVNI^+$ or a oscillates. Hence we have

PROPOSITION 3.4. (i) For each $a \in \operatorname{int} A^+$ there exists a neighborhood $\mathcal{N}$ of a, $\mathcal{N} \subset A^+$ such that for all $b \in \mathcal{N}$, $T \equiv \inf_{n\in\mathcal{N}} T(b) > 0$. There is a constant L in which for any a_1, $a_2 \in \mathcal{N}$ and any $t \in [0, T)$, $\|u(t, a_1) - u(t, a_2)\| \leqslant \|a_1 - a_2\|$.

(ii) For each $a \in A_0^+ \cap NI^+$ we can find a sequence b^n in A_0^+ for which $b^n \to a$ as $n \to \infty$ but $T(b^n) \equiv 0$ for all n and $T(a) = \infty$; that is, there is a global solution through a but no solution through b^n, $n = 1, 2, \ldots$.

(iii) Similarly, for each $a \in A_0^+$ for which $u(t, a)$ is a nonextendable solution we can find a sequence b^n in A_0^+ for which $b^n \to a$ as $n \to \infty$ but $T(b^n) \equiv 0$ for all n and $T(a)$ is finite.

Thus we don't have continuous dependence in A^+.

As we have seen if $u(t, a)$ is a global solution then $u(t, a) \to 0$ as $t \to \infty$. If $a \in A^+$ then $u(t, a)$ is a global solution if and only if $a \in NI^+$. For any $a \in A^+ \cup A^-$ we say $u(t, a)$ is a global attractor if $u(t, a)$ is a global solution and for each neighborhood $\mathcal{N} \in A^+ \cup A^-$ of a and for each $n \in \mathcal{N}$ for which $u(t, b)$ is a global solution we have $\|u(t, b) - u(t, a)\| \to 0$ as $t \to \infty$. For $a \in A^+ \cup A^-$ the only solution that is a global attractor is $u(t, 0) \equiv 0$.

We say $u(t, 0) \equiv 0$ is stable if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $\|u(t, b)\| < \varepsilon$ for all $t \in [0, T(b))$ whenever $b \in A^+ \cup A^-$ and $\|b\| < \delta$.

Hence $u(t, 0) \equiv 0$ is a stable global attractor or globally asymptotically stable. In finite dimensions it is usually assumed that the stable solution is in the interior of the domain. Then it is automatically true that solutions exist globally in a neighborhood of a stable solution, that is, $T(b) = \infty$. For general dynamical systems such as the one generated by (2.1) this property that $T(b) = \infty$ for b in any neighborhood of $\alpha = 0$ is not needed even if there also exists a C^1 Liapunov function $V(u)$ defined for $u \in A^+ \cup A^-$ satisfying

(a) $V(u)$ is positive definite and decrescent,

(b) $\dot{V}(u)$ is negative definite, where $\dot{V}(u(t))$ is the derivative of V along the solution through u for $t \in [0, T(\omega))$.

Under these conditions the origin is globally asymptotically stable (see [3] for further discussion in Banach spaces). Now in our case define

$$V(u) = \sum_{j=1}^{\infty} \frac{u_j^2}{j^2}.$$

Clearly $V(u)$ satisfies (a). Moreover for $u \in A^+ \cup A^-$

$$\sum_{j=1}^{\infty} \frac{-2u_j \|u^j\|}{j^2} \leqslant \dot{V}(u) \leqslant \sum_{j=1}^{\infty} \frac{-2u_j^2}{j^2}$$

Since $-\|u^j\| \leqslant -u_j$ for $u \in A^+$ $\dot{V}(u)$ satisfies (b). Notice the above does not require knowledge of the local or global existence of solutions.

To summarize we have seen that stability and asymptotic stability in a Banach space do not require either the local existence or global existence of solutions [4]. We have also seen that we can use Liapunov functions whose behavior along solutions does not require knowledge of those initial points yielding nonexistence, local existence, or global existence. Of course it would be sufficient that $V(u)$ satisfy (a) and (b) relative only to those points u for which $0 < T(u) \leqslant \infty$, but such a V might be quite difficult to construct.

We thus have

PROPOSITION 3.5. *Consider the set $A^+ \cup A^-$. Then*

(i) $u(t, 0) \equiv 0$ *is the only globally asymptotically stable solution,*

(ii) *there exists a C^1 Liapunov function $V(u)$ defined on $A^+ \cup A^-$, which is positive definite, decrescent, and $\dot{V}(u)$ is negative definite for all $u \in A^+ \cup A^-$ for $t \in [0, T(u))$.*

4. Example in A, Asymptotic Stability

The analysis involved with (2.1) in $A^+ \cup A^-$ carries over to A with minor modifications. Hence we shall include only the main ideas and let the reader fill in most of the details. Let $a \in A$ and we now define a decomposition, $\{I_i\}_{i=1}^N$, $0 \leqslant N \leqslant \infty$, of a. Let $K_0 = 1$ and define $|a_{\tau_1}| = \sup_{j \in K_0} |a_j|$. Then if $a_{\tau_1} > 0$ let $K_1 = \tau_1$; if $a_{\tau_1} < 0$ then let $j_1 = \tau_1$ and define K_1 as the next integer greater than j_1 for which $a_{K_1} > 0$. Define inductively for $i = 1, 2, \ldots, N$, $|a_{\tau_{i+1}}| = \sup_{j \geqslant K_i} |a_j|$. Then if $a_{\tau_{i+1}} > 0$ let $K_{i+1} = \tau_{i+1}$; if

$a_{\tau_{i+1}} < 0$ then let $j_{i+1} = \tau_{i+1}$ and define $j_{i+1} = \tau_{i+1}$ and define K_{i+1} to be the next integer greater than j_{i+1}. The process terminates when there exists an integer N, $0 \leqslant N \leqslant \infty$, such that K_{N+1} does not exist. Define $I_i = [K_{i-1}, K_i]$ $i = 1, 2, \ldots$, and then $[0, M) = \bigcup_{i=1}^{N} I_i$, where $M = K_N$. An interval I_i is said to be inadmissible if $K_i - K_{i-1} > 1$ and either there exists no $j_i \in I_i$ or there exists an integer $h_i \in (K_{i-1}, j_i)$ such that $a_{h_i} > 0$. If I_i is inadmissible then define $\tilde{I}_i \equiv I_i$ and let $\tilde{I}_i = [\tilde{K}_{i-1}, \tilde{K}_i]$ in which $1 \leqslant i \leqslant P$, where $P \leqslant \infty$. For completeness if $p = 0$ then all the intervals I_i are admissible. If p is infinite we say a oscillates. In those $\tilde{I}_i$ for which there exist no j_i we say a has upward slope and in those $\tilde{I}_i$ for which there exists j_i we say a has downward slope. We now measure the total upward slope and total downward slope. To this end let $\tilde{I}_{r_i}$ be those inadmissible intervals where a has upward slope and assume there exist P_1 of them, $0 \leqslant P_1 \leqslant \infty$, and let $\tilde{I}_{d_i}$ be those inadmissible intervals for which a has downward slope and assume there are P_2 of them, $0 \leqslant P_2 \leqslant \infty$. On each interval $\tilde{I}_{r_i}$, $a_{s_i} = \inf_{j \in \tilde{I}_{r_i}} \{a_j\}$ and let $m_{r_i}(a) = a_{\tilde{K}_{r_i}} / a_{s_i}$. Then define $m_r(a) = \sup_{1 \leqslant i \leqslant P_1} m_{r_i}(a)$ and if $P_1 = 0$ let $m_r(a) = 1$. Similarly on each interval $\tilde{I}_{d_i}$ let $a_{\tau_i} = \inf_{j \in \tilde{I}_{d_i},\, a_j > 0} \{a_j\}$ and let $m_{d_i}(a) = |a_{j_{r_i}}| / a_{\tau_i}$. Then define $m_d(a) = \sup_{1 \leqslant i \leqslant P_2} m_{d_i}(a)$. If $P_2 = 0$ let $m_d(a) = 1$. Finally, define $m(a) = \max\{m_r(a), m_d(a)\}$ and $m(a)$ has the same properties as that in Section 3. In particular if $m(a) = \infty$ we say a oscillates with unbounded upward slope ($m_r(a) = \infty$) and/or a oscillates with unbounded downward slope ($m_d(a) = \infty$). The behavior of solutions in $\tilde{I}_i$ is similar to that in the previous section. Namely in $\tilde{I}_{r_i}$, $u_{s_i}(t) = a_{s_i} + a_{\tilde{K}_{r_i}}(e^{-t} - 1)$ or $u_{\tilde{K}_{r_i}}(t) - u_{s_i}(t) \equiv u_{\tilde{K}_{r_i}}(0) - u_{s_i}(0) = a_{\tilde{K}_{r_i}} - a_{s_i}$ since $u_{\tilde{K}_{r_i}} = a_{\tilde{K}_{r_i}} e^{-t}$. In $\tilde{I}_{d_i}$ we have $u_{j_{r_i}}(t) = a_{j_{r_i}} e^{-t}$ and $\dot{u}_{\tau_i} = -|a_{j_{r_i}}| e^{-t}$, which implies $u_{\tau_i}(t) = a_{\tau_i} + |a_{j_{r_i}}|(e^{-t} - 1)$. Hence $|u_{j_{r_i}}(t)| - u_{\tau_i}(t) \equiv |a_{j_{r_i}}| - a_{\tau_i}$. Also as before if we define $T(a)$ to be the exit time for $u(t, a)$ then we find $T(a) = \ln(m(a)/(m(a) - 1))$. If I_i is admissible then as before let $m_i(a) = 1$ and thus if all the I_i are admissible then $m(a) = 1$ and thus $u(t, a)$ is a global solution. The converse is also true, that is, if $u(t, a)$ is a global solution then the decomposition of a contains no inadmissible intervals; for example, $|a_{i+1}| \leqslant |a_i|$ for all i. Also if $m(a) = \infty$ then $u(t, a)$ does not exist if and only if a oscillates with either unbounded upward or unbounded downward slope. Moreover if $1 < m(a) < \infty$, $m(u(t, a))$ is strictly increasing. Hence Proposition 3.2 carries over. If $u(t, a)$ is a global solution then $u(t, a) \to 0$ as $t \to \infty$; if $u(t, a)$ is a nonextendable solution then there exists $T(a) > 0$ such that $u(t, a) \in A$ for $t \in [0, T(a))$ but $\lim_{t \to T(a)} u(t) \notin A$. We shall characterize those points a for which $u(t, \alpha)$ is a global solution, nonextendable solution, or nonexistent solution in terms of admissible or inadmissible intervals or in terms of $m(a)$ without providing additional characterization such as a being nonincreasing, nondecreasing, eventual nonincreasing, etc., as is done in Proposition 3.1. In this way the existence of a global solution $u(t, a)$ is equivalent to the nonexistence of any inadmissible intervals, that is, $m(a)=1$; and the existence of a nonextendable solution is equivalent to a oscillating with unbounded upward slope or unbounded lower slope, that is, $m(a)=\infty$. The existence of a nonextendable solution is equivalent to a having at least one inadmissible interval and either a finite number of inadmissible intervals or an infinite number of inadmissible intervals in which a oscillates with either bounded upward slope or bounded downward slope (OB), that is, $1<m(a)<\infty$. So in this way a modification of Proposition 3.1 can be accomplished. The discussion on invariant sets presented in Proposition 3.3 essentially remains the same in the general case when $a \in A$. A similar characterization of invariant sets can be given in terms of the decomposition of a or in terms of $m(a)$. Namely, let $G=\{a: m(a)=1\}$, then G is the largest globally invariant set in A; that is, every other globally invariant set in A contained in G and G also has a characterization in terms of the decomposition of a. Similarly if we let $L=\{a\colon 1<m(a)<\infty\}$ then L is the largest locally invariant subset of A and it can be characterized in terms of a decomposition of a, that is, in terms of admissible as well ad inadmissible intervals. This characterizes Proposition 3.3 in the more general case. Again the solution $u(t,0)\equiv 0$ is the only globally asymptotically stable solution in A and the Liapunov function defined in A as $V(u)=\sum_{j=1}^{\infty}(u_j^2/j)$ satisfies the conditions in Proposition 3.5. It remains only to discuss Proposition 3.4 for $\alpha=A$. Indeed for $a \in \operatorname{int} Aj$ the same computations preceding Proposition 3.4 show that $u(t,a)$ is Lipschitz continuous with respect to a. Moreover for each neighborhood $\mathcal{N}$ of a there exists a time $T(\mathcal{N})$ such that for each $b \in \mathcal{N}$, $u(t,b)$ exists on $[0,T]$. Also we can show that continuous dependence arguments do not hold in A_0. Thus Proposition 3.4 can also be extended. We now summarize the extension to $a \in A$ of Propositions 3.1–3.5 in the following:

PROPOSITION 4.1. I. *Let $a \in \operatorname{int} A$, then either $u(t,a)$ is a global solution or a nonextendable solution and $u(t,a)$ is locally Lipschitz in a.*

A. *Let $u(t,a)$ be a global solution.*

1. *The following are equivalent:*

   (i) *$u(t,a)$ is a global solution,*

   (ii) *$m(a)=1$ (and then $m(u(t,a))\equiv 1$ for $t \geqslant 0$),*

   (iii) *there are no inadmissible intervals in the decomposition of a,*

   (iv) *$u(t,a)\to 0$ as $t\to\infty$.*

2. *$G=\{a \in \operatorname{int} A\colon m(a)=1\}$ is the largest globally invariant set in* int A *(any other globally invariant set in* int A *is a subset of G).*

B. *Let $u(t, a)$ be a nonextendable solution.*

1. *Then the following are equivalent:*

(i) *$u(t, a)$ is a nonextendable solution,*
(ii) *$1 < m(a) < \infty$ and $m(u(t, a))$ is strictly increasing in t,*

(iii) *a oscillates with both bounded upward slope and bounded downward slope or oscillates finitely or doesn't oscillate and is eventually nondecreasing.*

2. *The exit time $T(a) = \ln(m(a)/(m(a) - 1))$ and $m(u(t, a))$ is a strictly increasing function for $t \in [0, T(a))$.*

3. *$L = \{a \in \operatorname{int} A : 1 < m(a) < \infty\}$ is the largest locally invariant set in* int A *(any other locally invariant set in* int A *is a subset of L).*

II. *Let $a \in A_0 = \partial A \cap A$.*

1. *Then either:*

a. *$u(t, a)$ is a global solution for whice* I.A.1 *and* I.A.2 *hold with* int A *replaced by A, or*

b. *$u(t, a)$ is a nonextendable solution for which* I.B.1 *and* I.B.2 *hold with* int A *replaced by A, or*

c. *$u(t, a)$ does not exist which is equivalent to $m(a) = \infty$, that is, a oscillates with either unbounded upward or downward slope.*

2. *There is no continuous dependence of solutions on initial conditions.*

III. *The solution $u(t, 0) \equiv 0$ is globally asymptotically stable. There is a Liapunov function $V(u) = \sum_{j=1}^{\infty} (u_j^2/j^2)$ which is globally positive definite and decrescent and $\dot{V}(u)$ is globally negative definite.*

In concluding this section we point out that variations of Example 2.1 can be constructed to achieve particular purposes. We shall confine our attention to A^+ and still assume the origin is globally asymptotically stable. Consider the differential equation defined in A^+ given by

$$\begin{aligned} \dot{u}_j &= u_j(1 - b_j) - \|u^j\|, \\ u(0) &= a, \end{aligned} \tag{4.1$_b$}$$

where b is any element in A_0^+ such that $\|b\| \geqslant 1$, $b \neq 0$. The continuity properties of the right-hand sign are clearly the same as $h(u)$. The analysis of $(4.1)_b$ is similar to that of (3.1), where $b = 1$ (relative to A^+) in that the qualitative behavior of solutions can be ascertained by analysis of the decomposition of a. The role of b_j is to influence the rate at which solutions decay as well as the exit time for the nonextendable solutions. In other words, b acts as a type of control parameter. We leave it to the reader to show for $(4.1)_b$ that $m(a, b) \equiv m(a)$ and $T(a, b) \equiv T(a)$.

5. Stability

In this section we shall give an example, in fact $(4.1)_0$ $(b=0)$, for which the origin is globally stable but not an attractor. As nothing is lost we shall consider the differential equation in A^+. Consider for $a \in A^+$ the equation

$$\dot{u}_j = u_j - \|u^j\|$$
$$u_j(0) = 0. \tag{5.1}$$

Using the same notation as in Section 3 we consider a decomposition of a, $\{I_i\}$, where $\{\tilde{I}_i\}_{i=1}^N$, $1 \leqslant N \leqslant \infty$, are the set of inadmissible intervals. Recall that each I_i, $\tilde{I}_i$ may be bounded or unbounded. Recalling $\tilde{I}_i = [\tilde{K}_{i-1}, \tilde{K}_i]$, then $u_{K_i} = u_{\tilde{K}_i} - \|u^{\tilde{K}_i}\| = u_{\tilde{K}_i} - u_{\tilde{K}_i} = 0$, that is, $u_{\tilde{K}_i}(t) \equiv u_{\tilde{K}_i}(0) = a_{\tilde{K}_i}$ for all $t \geqslant 0$. Now let $u(t, a)$ be nonextendable. In this case there exists a $i_1 \in (\tilde{K}_{i_1-1}, \tilde{K}_i)$, where $a_{i_1} < a_{\tilde{K}_{i_1}}$. This implies

$$\begin{aligned} u_{i_1} &= u_{i_1} - \|u^{\tilde{K}_{i_1}}\| \\ &= u_{i_1} - u^{\tilde{K}_{i_1}} \\ &= u_{i_1} - a_{\tilde{K}_{i_1}} \end{aligned}$$

for as long as u_{i_1} exists. Then

$$u_{i_1}(t) = a_{i_1} e^t - a_{\tilde{K}_{i_1}}(e^t - 1),$$

or

$$u_{i_1}(t) = (a_{i_1} - a_{\tilde{K}_{i_1}}) e^t + a_{\tilde{K}_{i_1}}.$$

Hence the exit time $T_{i_1}(a)$ on $\tilde{I}_{i_1}$ is given by solving

$$0 = (a_{i_1} - a_{\tilde{K}_{i_1}}) e^t + a_{\tilde{K}_{i_1}},$$

which implies

$$T_{i_1}(a) = \ln\left(\frac{a_{\tilde{K}_{i_1}}}{a_{\tilde{K}_{i_1}} - a_{i_1}}\right).$$

Thus the exit time, $T(a)$, is given by $T(a) = \inf_{i \in \tilde{I}_i} T_i(a)$.

We define the measure of variation of a in $\tilde{I}_{i_1}$, $m_{i_1}(a)$, exactly as before and $m(a)$, the measure of variation of a, as $m(a) = \sup_{1 \leqslant i \leqslant N} m_i(a)$. As before $T(a)=\ln(m(a)/(m(a)-1))$ and $m(u(t,a))$ is a strictly increasing function in t on $[0, a)$ and increases more rapidly than in (3.1). In fact the only essential difference between (5.1) and (3.1) is that all global solutions in (5.1) are constants. Then Proposition 3.5 changes. In this case define the Liapunov function as before, $V(u)=\sum_{j=1}^{\infty}(u_j/j^2)$. Then

$$\dot{V}(u)=2\sum_{j=1}^{\infty}\frac{u_j\dot{u}_j}{j^2}=2\sum_{j=1}^{\infty}\frac{u_j(u_j-\|u^j\|)}{j^2}=2\sum_{j=1}^{\infty}\frac{u_j^2-u_j\|u^j\|}{j^2}.$$

Thus $\dot{V}(u)\leqslant 0$ and $\dot{V}(u)=0$ if and only if $\|u^j(t)\|\equiv u_j(t)$, which holds if and only if $u(t,a)\equiv a$. Thus $V(u)$ satisfies the conditions that the solution $u(t,a)\equiv 0$ is stable. We also see that the largest global invariant set in A^+ is the set of constants. The locally invariant sets in A^+ are the same as that given in Proposition 3.3. Proposition 3.4 remains exactly the same. We then obtain the following

PROPOSITION 5.1. *Consider the system* (5.1) *and let* $a\in A^+$. *Then Propositions* 3.1 *through* 3.5 *are applicable to* (5.1) *with the only change being that all global solutions are constant, the solution* $u(t,a)\equiv 0$ *is stable but not an attractor, and the largest globally invariant set is given by* $D=\{a\in A^+\colon a \text{ is a constant}\}$.

Of course we can extend (5.1) to all of A and obtain results analogous to Proposition 4.1. In concluding this section we point out the example in A^+ given by

$$\begin{aligned}\dot{u}_j&=\|u^j\|-u_j\\ u(0)&=a.\end{aligned}\tag{5.2}$$

In this case all solutions are global and if we decompose a as before and let $\tilde{I}_i=[\tilde{K}_{i-1}, \tilde{K}_i]$, $i=1,\ldots,N$, be the inadmissible intervals then for each integer $n\in(\tilde{K}_{i-1}, \tilde{K}_i)$ we have $u_{\tilde{K}_i}(t)\equiv a_{\tilde{K}_i}$ and $u_n(t)\to a_{\tilde{K}_i}$ as $t\to\infty$ and $u_n(t)$ is strictly increasing in t. The solution $u(t,a)\equiv 0$ is stable.

6. CONCLUDING REMARKS

Notice that since $\overline{\operatorname{int} A^+}=E^+$ we find that our differential equations are defined on a "large" set in the topological sense. Also the measure of B^+ is zero, which is the complement of A^+.

When $a\in A^+$ we may consider (3.1) as representing a process consisting of a large number of compartments containing some "resource." The time of depletion of the resource in a compartment obeys a first-order law which is based on the amount of upward variation of the initial condition

$a = \{a_i\}_{i=1}^{\infty}$, that is, $m(a)$. We think of a_i as the amount of resource in compartment i. The process terminates when one compartment depletes all of its resources. We can attempt to extend the life of a process by either altering the initial state so that we minimize the variation of a or by modifying the dynamics through the introduction of a parameter as in $(4.1)_b$. Since these processes are affected by external disturbances then since (3.1) has associated with it a Liapunov function it is easy to determine the admissible perturbations that will preserve the asymptotic stability. It would be interesting to see what effects these disturbance have on $m(u(t, a))$ for arbitrary a.

We point out for (3.1) that the $\partial A^+ \cap A^+$ consists of points through which quite different types of behavior of solutions occur: nonexistence, nonextendability, global existence (of solutions), no continuous dependence of solutions on initial conditions. In the int A^+, solutions always exist in which some are nonextendable while others exist globally. Some of these properties do not hold in the finite-dimensional version of (3.1) particularly since $a \in R^n$ implies $m(a) < \infty$. Hence local existence and continuous dependence are always true. Also our stability analysis carries over to the finite-dimensional version in that the origin is on the boundary of $E^n = E \cap R^n$ and so we need to consider the more generalized notion [4].

In conclusion, generalization of the ideas that were used in constructing and analyzing (3.1) can be formulated. We can allow A to be a general cone in E. In the decomposition of a we essentially used the fact that the sets for which $m(a) = 1$, or $1 < m(a) < \infty$, or $m(a) = \infty$ are each invariant. This can also be generalized by defining an abstract version of measure relative to a decomposition.

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